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LETTER TO THE EDITOR

On the motion equations of a spinning fluid

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Abstract. Using super-Hamiltonian formalism, we derive the motion equations of the adiabatic dynamics of a spinning fluid.

The purpose of this letter is to derive the motion equations of a non-relativistic fluid whose particles carry spinning degrees of freedom. The method of derivation is based on the Hamiltonian formalism.

We begin with the particle picture. Here the Poisson bracket (PB) has the form [1]

$$\{F, G\} = \text{canonical PB} + \text{Grassmann PB} \tag{1}$$

$$\text{canonical PB} = \frac{\partial F}{\partial x_i^s}\frac{\partial G}{\partial p_i^s} - \frac{\partial F}{\partial p_i^s}\frac{\partial G}{\partial x_i^s} \tag{1a}$$

$$\text{Grassmann PB} = \frac{F\overleftarrow{\partial}}{\partial \xi_\alpha^s}\frac{\overrightarrow{\partial} G}{\partial \xi_\alpha^s} \tag{1b}$$

where: $1 \leqslant i \leqslant n$; $1 \leqslant \alpha \leqslant N$; $1 \leqslant s \leqslant S$; n $(=3)$ is the dimension of space; S is the number of particles; N $(=3)$ is the number of components which the odd Grassmann spin variables $\{\xi^s\}$ carry; we sum on repeated indices unless warned not to; $\overleftarrow{\partial}/\partial\xi$ and $\overrightarrow{\partial}/\partial\xi$ $(=\partial/\partial\xi)$ are the right and left partial ξ-derivatives respectively.

The PB (1) is a sum of canonical PB $(1a)$ and the Grassmann PB $(1b)$ in $\oplus_s\Lambda(N)$. The latter bracket, for each fixed s, can be uniquely characterised by the property that the map

$$R_{\alpha\beta} = \xi_\alpha\xi_\beta \tag{2}$$

is canonical into the Lie–Poisson bracket on the dual space of the Lie algebra $\mathscr{G} :=$ so(N):

$$\{R_{\alpha\beta}, R_{\mu\nu}\} = R_{\alpha\nu}\delta_{\beta\mu} - R_{\alpha\mu}\delta_{\beta\nu} + R_{\beta\mu}\delta_{\alpha\nu} - R_{\beta\nu}\delta_{\alpha\mu}. \tag{3}$$

Imagine now that S is large, so that we can treat the motion of our collection of particles as a flow. Set

$$M_i(x) := p_i^s\delta(x - x_s) \tag{4}$$

$$\rho(x) := m^s\delta(x - x_s) \tag{5}$$

$$\omega_\alpha(x) := \sqrt{m^s}\,\xi_\alpha^s\delta(x - x_s) \tag{6}$$

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where m^s is the mass of the sth particle. Using the method of Bialynicki-Birula and Hubbard [2], we convert the particle PB (1) into the following field PB:

$$\{M_i(x), M_j(x')\} = [M_j(x)\partial_i + \partial_j M_i(x)]\delta(x-x') \tag{7a}$$

$$\{M_i(x), \rho(x')\} = \rho(x)\partial_i\delta(x-x') \tag{7b}$$

$$\{M_i(x), \omega_\alpha(x')\} = \omega_\alpha(x)\partial_i\delta(x-x') \tag{7c}$$

$$\{\omega_\alpha(x), \omega_\beta(x')\} = \lambda\delta_{\alpha\beta}\rho(x)\delta(x-x') \tag{7d}$$

where $\partial_i = \partial/\partial x_i$ and $\lambda = 1$ is inserted into (7d) for future use.

Physics having served its purpose, we now revert to the mathematical language. The PB (7) correspond to the following (super) Hamiltonian matrix† $\mathbf{B}_1$:

$$\mathbf{B}_1 = \begin{array}{cc} & \begin{array}{ccc} M_j & \rho & \omega_\beta \end{array} \\ \begin{array}{c} M_i \\ \rho \\ \omega_\alpha \end{array} & \begin{pmatrix} M_j\partial_i + \partial_j M_i & \rho\partial_i & \omega_\beta\partial_i \\ \partial_j\rho & 0 & 0 \\ \partial_j\omega_\alpha & 0 & \lambda\rho\delta_{\alpha\beta} \end{pmatrix} \end{array}. \tag{8}$$

Being linear in the field variables, the Hamiltonian matrix $\mathbf{B}_1$ corresponds [3] to a Lie superalgebra, $\mathscr{L}(\lambda)$; the commutator in the latter is‡

$$\left[\begin{pmatrix} X^1 \\ f^1 \\ \boldsymbol{\gamma}^1 \end{pmatrix}, \begin{pmatrix} X^2 \\ f^2 \\ \boldsymbol{\gamma}^2 \end{pmatrix}\right] = \begin{pmatrix} [X^1, X^2] \\ X^1(f^2) - X^2(f^1) - \lambda\boldsymbol{\gamma}^1 \cdot \boldsymbol{\gamma}^2 \\ X^1(\boldsymbol{\gamma}^2) - X^2(\boldsymbol{\gamma}^1) \end{pmatrix} \tag{9}$$

where: X^1, $X^2 \in D_n(K) := \{$the Lie algebra of even vector fields (derivatives) in a commutative superalgebra K with n commuting even derivatives $\partial_1, \ldots, \partial_n\}$; $f^1, f^2 \in K_{\bar{0}}$ (even elements of the ring K); $\boldsymbol{\gamma}^1$, $\boldsymbol{\gamma}^2 \in (K_{\bar{1}})^N$ (N-component vectors of odd elements of the ring K) and

$$X(\cdot) := X_i(\cdot)_{,i} := X_i\partial_i(\cdot). \tag{10}$$

The role of λ is clear from (9): for $\lambda = 0$, the Lie superalgebra $\mathscr{L}(0)$ becomes the Abelian semidirect sum

$$\mathscr{L}(0) = D_n \propto [K_{\bar{0}} \oplus (K_{\bar{1}})^N] \tag{11}$$

while this is not the case for $\lambda \neq 0$. (Notice that $\mathscr{L}(\lambda_1) \approx \mathscr{L}(\lambda_2)$ when $\lambda_1\lambda_2 \neq 0$.)

† The Hamiltonian matrix $\mathbf{B}$ is defined so that for a vector of basic variables A and Hamiltonian (density) H, $\dot{A} = \mathbf{B} \cdot \delta H/\delta A$. For a particle system with dynamical variables q_i and p_j,

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

‡ The commutator is defined in terms of the Hamiltonian matrix $\mathbf{B}_1$ by

$$-\mathbf{B}_1 \begin{pmatrix} X^1 \\ f^1 \\ \boldsymbol{\gamma}^1 \end{pmatrix}^t \begin{pmatrix} X^2 \\ f^2 \\ \boldsymbol{\gamma}^2 \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{M} \\ \rho \\ \omega \end{pmatrix}^t \left[\begin{pmatrix} X^1 \\ f^1 \\ \boldsymbol{\gamma}^1 \end{pmatrix}, \begin{pmatrix} X^2 \\ f^2 \\ \boldsymbol{\gamma}^2 \end{pmatrix}\right]$$

where $\sim$ stands for the equality modulo divergences.

The $\{\boldsymbol{M}, \rho\}$ peice of the Hamiltonian matrix $\mathbf{B}_1$ corresponds to the case of the barostropic fluid dynamics; if we wish to take the entropy into account we simply $\boldsymbol{\omega}$-extend the corresponding adiabatic Hamiltonian matrix. The result is

$$B_2 = \begin{array}{c} \\ M_i \\ \rho \\ \eta \\ \omega_\alpha \end{array} \begin{array}{c} \begin{array}{cccc} M_j & \rho & \eta & \omega_\beta \end{array} \\ \begin{pmatrix} M_j\partial_i + \partial_j M_i & \rho\partial_i & -\eta_{,i} & \omega_\beta \partial_i \\ \partial_j \rho & 0 & 0 & 0 \\ \eta_{,j} & 0 & 0 & 0 \\ \partial_j \omega_\alpha & 0 & 0 & \lambda\rho\delta_{\alpha\beta} \end{pmatrix} \end{array} \tag{12}$$

where η is the specific entropy. The particle map (2) has the following field analogue: the map

$$L_{\alpha\beta} = \rho^{-1}\omega_\alpha\omega_\beta \tag{13}$$

is canonical between the super-Hamiltonian matrix $\mathbf{B}_2$ (12) and the Hamiltonian matrix $\mathbf{B}_3$:

$$B_3 = \begin{array}{c} \\ M_i \\ \rho \\ \eta \\ L_{\alpha\beta} \end{array} \begin{array}{c} \begin{array}{cccc} M_j & \rho & \eta & L_{\mu\nu} \end{array} \\ \begin{pmatrix} M_j\partial_i + \partial_j M_i & \rho\partial_i & -\eta_{,i} & L_{\mu\nu}\partial_i \\ \partial_j \rho & 0 & 0 & 0 \\ \eta_{,j} & 0 & 0 & 0 \\ \partial_j L_{\alpha\beta} & 0 & 0 & \lambda\{L_{\alpha\beta}, L_{\mu\nu}\} \end{pmatrix} \end{array} \tag{14}$$

where $\{L_{\alpha\beta}, L_{\mu\nu}\}$ is given by the formula (3) with the variables R and L interchanged. Thus, apart from the $\{\rho, \eta\}$ piece, the Hamiltonian matrix $\mathbf{B}_3$ corresponds to the current Lie algebra

$$D_n \propto (\mathscr{G} \otimes K) \tag{15}$$

and the full Hamiltonian matrix $\mathbf{B}_3$ corresponds to the semidirect sum Lie algebra

$$D_n \propto [K \oplus \Lambda^n \oplus (\mathscr{G} \otimes K)]. \tag{16}$$

We now are in a position to derive the motion equations. Having found the Hamiltonian structure(s), we need only to specify the Hamiltonian function, i.e. the total energy. Since we are dealing with the non-relativistic case, we take the total energy, as in adiabatic fluid dynamics, to be the sum of kinetic and potential energies

$$H = \frac{\boldsymbol{M}^2}{2\rho} + \rho e \tag{17}$$

where e is the specific energy (inernal + external):

$$e = e(\rho, \eta, L, x). \tag{18}$$

Thence, we obtain the motion equations

$$-M_{i,t} = (\rho^{-1} M_i M_j + \delta_{ij} P)_{,j} + \rho \frac{\partial e}{\partial x_i} \tag{19a}$$

$$-\rho_{,t} = \operatorname{div}(\boldsymbol{M}) \tag{19b}$$

$$-\eta_{,t} = \rho^{-1}\eta_{,j} M_j \tag{19c}$$

$$-\omega_{\alpha,t} = \operatorname{div}(\omega_\alpha \rho^{-1} \boldsymbol{M}) + \lambda\rho^2\omega_\alpha \frac{\partial e}{\partial \omega_\alpha} \qquad \text{(no sum on } \alpha) \tag{19d}$$

where

$$P := \left(\rho \frac{\partial}{\partial \rho} + \omega_\beta \frac{\partial}{\partial \omega_\beta} - 1\right)(\rho e) \tag{20}$$

is the pressure. In the case that the even variables $L_{\alpha\beta}$ are chosen instead of the odd ones ω_α, one has

$$-L_{\alpha\beta,t} = \operatorname{div}(L_{\alpha\beta}\rho^{-1}\boldsymbol{M}) + \lambda\rho\{L_{\alpha\beta}, L_{\mu\nu}\}\frac{\partial e}{\partial L_{\mu\nu}} \tag{21}$$

with the pressure function P

$$P := \left(\rho \frac{\partial}{\partial \rho} + L_{\mu\nu} \frac{\partial}{\partial L_{\mu\nu}} - 1\right)(\rho e). \tag{22}$$

Remark. From formulae (19d) and (21) we see that the spinning variables ω and L are *not* frozen-in.

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